THE MATHEMATICS TEACHER

EDITED BY

W. H. METZLER

ASSOCIATED WITH

EUGENE R. SMITH HARRY D. GAYLORD

MAURICE J. BABB WILLIAM E. BRECKENRIDGE

VOLUME VII SEPTEMBER, 1914 NUMBER 1

THE MATHEMATICAL PESSIMIST.

By W. R. Ransom.

By the mathematical pessimist I mean the discouraged student, the boy (or girl) who has arrived at the conclusion that mathematicians, like poets, are born and not made, and that none was born on his birthday. He believes himself to be what Klein has called an "amathematician." We may quote the very words of the pessimist: you recognize the species when he refers to his geometry or his algebra—oh especially his algebra!—as "this stuff." Such a one was the writer of a charming Christmas story who began it with the statement that she could not bear the abbreviation "Xmas" because, she said, "X always seemed to me to stand for what was hard and difficult, unknown and unknowable."

I do not mean the happy blunderer who cannot get his answers right, the wild Indian whose tomahawk removes the radical from $\sqrt{a^2+x^2}$ with a swoop that takes the exponents off with it;—nor the young sinner who on meeting the equation $\sin x = 0$ gets rid of his sin by dividing it out, and for penance must lose the π to which he was entitled;—nor the girl who remonstrated with me when I agreed that $abc2x = 2abcx$ but held that $\sin 2x$ does not equal $2 \sin x$. "Why," said she, "I thought that when you learned a thing in mathematics it was always so."

No these persons are not your true pessimists. They are in their way good mathematicians, but not very profound and some-

what blundering. They forget the rules and reason by analogies which no doubt are more convincing to them at the moment than genuinely rigorous "epsilon-arguments" are to many a graduate student. Such persons are much more numerous than the pessimists and equally deserving of having papers written about them. I did not write about them for this reason: I know all about the bunglers and I should have so authoritatively settled their problems in a few pages that the subject would be finally closed. About the treatment and cure of pessimism, however, I know so little that I must infallibly say things so wrong as to provoke discussion. In this way I have hoped that this paper might lead to a profitable discussion from which I might learn many things that I am very anxious to know.

The pessimist is strong in his conviction. May he not sometimes be right? Maybe some people have a limit of mathematical power, as certainly all have of physical power. The limit may be at the first chapter of the algebra, or the line may be drawn still lower. There may be no such line: it may be a wholly imaginary line and yet to the poor pessimist be as terrible and grievous as the Equator to the little girl who defined it as a "menagery lion running around the earth."

We have all seen so often the wonderful change of mind that comes in school with a change of heart, the transformation of the hopeless student into the successful one, that most of us are reluctant to believe that any student is hopeless. In fact we are Optimistic about the worst of the Pessimists.

Professor Felix Klein says that the question whether there *are* "amathematicians" must be left to the psychologists to determine. Professor Julian Coolidge, answering an advocate of "practical mathematics only," said that we all study mathematics in the schools "because our minds are incurably mathematical." At the Albany meeting last November, Commissioner C. F. Wheelock gave statistics proving that low averages in mathematics are due to poor teaching rather than to the difficulties of the subject and he seemed to conclude that there are no cases of "amathematicism." Professor M. J. Babb agreed with this conclusion and gave excellent advice on improving both the teaching and the selection of topics, and on training the students to unite "chalk and talk." So far the authorities disagree with the pessimist's belief in his "amathematicism."

I am inclined to think the question still an open one. It may indeed turn out that none are by nature devoid of geometrical and analytical powers. Yet I should still fear that some pupils may grow into habits of thought which make the further cultivation of mathematics during their few schoolable years a practical impossibility.

We do not need to settle the question. Experience shows that some pessimists are curable. What then is the cure?

The personality of the teacher is the most potent aid. If a pupil believes that the teacher is thoroughly interested in his subject and thoroughly interested in the pupil, a link of the strongest kind has been made between the subject and the pupil. "Love me, love my dog" is good doctrine, even if the dog's name is Algebra. There is not much I can *say* about this: it is something to *be*. And I suppose we are all trying to be it as hard as we can, to see with our pupils' eyes, to feel with their hearts, that they may see with our eyes, feel with our hearts, and think with minds not perversely different from our own.

Professor Frank E. Seavey, of the English department at Tufts College, a man of rare ability in winning the confidence of college boys, tells me that we mathematics teachers, and even such of our students as get *A*'s and *B*'s cannot understand how things look to the fellow near the firing line, the student who works hard and gets a bare pass or fails.

The mathematical way of looking at things is abhorrent to the good sense of some otherwise minded folks. The proof for the non-existence of a maximum integer which begins "consider all possible things" provokes laughter. Such dicta as that zero divided by zero is not clearly equal to one, that there is no tangent of 90°, that a curve without width can fill up an area, that the numbers $+1$ and -1 are just as deserving of the epithet *imaginary* (in its non-technical sense) as the number $\sqrt{-1}$, that a line segment may have a last point but no next to the last point, that there are as many points in an inch of line as in a mile,—these and many other paradoxes are rich and satisfying food to your pure mathematician, but caviar to the general. And to the pessimist? The complacency with which we swallow such "absurdities" is the clearest proof that higher mathematics is a delusion.

I have had long and earnest arguments with mature and gifted gentlemen, who had stood by no means low in their college mathematics,—with one because he would not admit the modern geometer's right to use the phrase "point at infinity" when clearely there is no such place as infinity, and moreover the thing referred to is not a point at all; with another because he would not allow the logician to translate the words "some lawyers are honest" by symbols that would exclude what all the world knows is implied, that *some are not!**

Some men of great force fail to see anything in important branches of intellectual theory. Some condemn philosophy, some theology, some political economy as delusions. Sir William Hamilton's opinion that the study of mathematics lacks most of the qualifications we should claim for it is warmly endorsed by a department head in the Massachusetts Institute of Technology. Commissioner David Snedden would greatly reduce the teaching of mathematics in our high schools. If mature and thoughtful men may so quarrel with us over fundamentals, need we doubt that Professor Seavey is right in denying us the entrée into the minds of some of our students?

Perhaps it is well for us to be more humble in our attitude toward those who scorn our science. Professor DuBois, of Atlanta, fresh from the Congress of Races at London in 1911, told us that the most significant note at that great meeting was this appeal to the white men from the "lesser breeds without the law": "When you come to our lands to teach us the wonderful things that science has done for your civilization, do not forget that we too have a civilization or our own, developed to fit our needs, that some of our ways are better for us than your ways, that some of our thoughts are more worthy than yours. Do not insist on teaching us only: learn of us too. And let us meet not as lord and vassal, but as friends, ready to take as well as give, each receiving the other's best, and yielding honorably to each other."

Is not the teacher always in danger of playing white man to the little heathen in his classes? Do you know the school where the motto of the mathematics department is that line from Kip-

* The Mate got even with his Captain by entering once in the log book: "Captain Evans was sober all day to-day!"

ling's Mandalay, "Lord, what *do* they understand?" How much better the attitude of Chaucer's Clerk: "and gladly wolde he lerne and gladly teche."

Enumerating the sorts of men whose thoughts go wrong, Locke says: "The third sort is those who readily and sincerely follow reason, but for want of having that which one may call large, sound, roundabout sense, have not a full view of all that relates to the question. . . . They have a pretty traffic with known correspondents in some little creek . . . but will not venture out into the great ocean of knowledge."

This description is applicable too often to the teacher of algebra, who usually knows the algebraic routine perfectly but has no unbookish nor even other-bookish applications connected with it, no interests in the broad world of science through which runs his narrow algebraic creek.

It is only by continuing to be a student that one can sympathize with students. The teacher should spend less time in correcting papers and more in study. Master the elements of many new subjects. "Thou that teachest another, teachest thou not thyself?" Every winter attack some substantial study. A good book (not too long) to dig it out of, and a fellow teacher to vie with one and keep one hard at it are quite necessary.

It may be that some teachers do not realize what a field there is for this sort of work, or what fun and profit it yields. I will suggest some books, not beyond the abilities and well connected with the interests of the average high school teacher of mathematics, which, besides offering such obstacles to a too easy reading as are importatn for the purpose, will richly reward the labor spent upon them.*

Worthen: Dynamics of Rotation, Longmans.
Whitehead: Introduction to Mathematics, Holt.
Evans: Teaching of Mathematics, Houghton Mifflin.
Young: Fundamental Concepts of Algebra and Geometry, Macmillan.
Bonola: Non Euclidean Geometry, Open Court.
Manning: Irrational Numbers, Wiley.
Fine: Number System of Algebra, Leach Shewell and Sanborn.
Coffin: Vector Analysis, Wiley.

* See also *The Teachers College Bulletin* No. 11, Second Series, Jan. 28, 1911: Bibliography of books on mathematics suitable for a high school library.

Risteen: Molecules and the Molecular Hypothesis, Ginn.
Airy: Undulatory Theory of Light, Macmillan.
Poincaré: Science and Hypothesis, Scribner.
Klein: Famous Problems in Elementary Geometry, Ginn.
Hanus: Determinants, Ginn.
Johnson: Curve Tracing, Wiley.
Young: Monographs on Modern Mathematics, Longmans.
Manning: Non Euclidean Geometry, Ginn.
Richardson and Ramsay: Modern Plane Geometry, Macmillan.

Moreover, teachers who have had French or German but do not think they have a reading knowledge of either will do well to undertake one of the excellent small books in those languages. These should be read aloud, not translated except as a last resort in occasional paragraphs. Many of our teachers believe they are confined to English as earnestly as the pessimist believes he is shut out of higher mathematics. Such will profit greatly by the struggle and will learn with surprise how easy it is to convert a poor literary knowledge of the foreign tongue into a fair reading knowledge of mathematical French or German. Begin with a book on a very elementary subject like Plane Geometry, or beginners, algebra, where there are plenty of diagrams or equations to help out and where the mathematical vocabulary is developed as you go along. For instance:

Mahler: Ebene Geometrie, Nr. 41, Sammlung Göschen.
Schubert: Arithmetik und Algebra, Nr. 48, Sammlung Göschen.
Lemoine: Geometrographie, No. 18 in Scientia, Gauthier-Villars.

As the second venture in French try

Laurent: La Geometrie Analytique General, A. Hermann.
Couturat: L'Algebre de la Logique, No. 24 in Scientia.
Barbarin: Geometrie Non Euclidienne, No. 15 in Scientia.

Further suggestions in German are

Böger: Elemente der Geometrie der Lage, G. J. Göschen.
Doehlemann: Projektive Geometrie, Nr. 72, Sammlung Göschen.
Beutel: Algebraische Kurven, Nr. 435–6, Sammlung Göschen.
Valentiner: Vektoranalysis, Nr. 354, Sammlung Göschen.
Jaeger: Theoretische Physik, Nr. 76–7–8, and 374, Sam. Göschen.

The Sammlung Göschen is a remarkable collection of small books, written in a masterly manner, costing only twenty-five cents apiece, yet fully equal to English books costing six times as much. B. G. Teubner is getting out a similar series. The

Scientia series of Gauthier-Villars, costing forty cents each, includes other interesting books not noted above.*

It seems to me a misfortune that our own countrymen do not produce more *small* books on advanced mathematical subjects, distinctly intended for those who are not specialists. The lack of such books lends color to the prevailing belief that mathematics as a sport is like football and not to be indulged in after graduation except by professionals.

Subjects of live interest and great fascination like projective geometry, non-Euclidean geometry, logical algebra, elementary function theory, infinite assemblages, geometrical transformations, deserve to be treated not too profoundly, in small volumes intended primarily for persons with the average ability of the mathematics teacher in the small high school. Our first rate mathematicians are all more eager to explore remote corners of the field of knowledge. The building of easier roads through already explored country, with occasional far-seeing glimpses into new country, such as French writers especially give us, seems to me a task worthy the mettle of our best men. I regret that the *Annals of Mathematics* has abandoned its plan of giving occasional papers in which old discoveries were made more accessible rather than new theories disclosed.

Such a book ought not to be as difficult reading as (say) Townsend's translation of Hilbert's "Fundamentals of Geometry," or some of the Monographs in Young's collection; but it should be closely reasoned, rich in information, and not so abstract in point of view as to repel the class of readers of whom I have been speaking. The knowledge that people who do not have to are reading mathematical books is a check upon the crop of pessimists. The teacher, honestly laboring over such books, frequently baffled but usually triumphant, may hope to keep in truer sympathy with his less proficient pupils.

* Professor J. T. Rorer adds these from an Italian series costing about thirty-five cents each:

Pincherle: Algebra Complementare, Parts I. and II. and Esercizi.
Pincherle: Geometria pura elementare.
Pincherle: Geometria pura elementare (Esercizi sulla).
Pincherle: Geometria metrica e trigonometria.
Alasia: Geometria e trigonometria della sfera.
Rossotti: Formularis Scolastico di Elementar Matematica.
Pascal: Determinanti e applicazioni.

By the introduction of more *practical* problems into mathematics courses much is being done to stimulate the interest and reduce the pessimism of a large class of students. Like all good things the use of practical problems is apt to be overdone when first taken up. Some of the problems which are most practical at the glazier's, the tin-smith's, or the naval architect's, prove to be the most impractical in the mathematics classroom. A practical problem is one which does not have to be lugged in and explained but one which slides up and irresistibly confronts you demanding a solution. I am a believer in practical problems, but my definition of them has changed since I first began collecting them. I now think that proving the rule for casting out nines, or finding what is the conclusion that can properly be drawn from the premises

No cat has two tails
One cat has one more tail than no cat

and interesting puzzles, paradoxes, and fallacies are quite as practical in the classroom as steam-fitter's difficulties.

I suppose that most of our pessimists become such during the algebra course. If our algebra is too abstract (and I believe it is) I think the remedy lies outside the algebra classroom as well as within. Let me speak of the value of casual algebra.

Other teachers should not wait for the algebra class to work up a thorough lay-out of algebraic usage, but should make use at will of simple algebraic equations and formulas as early as desired. I was taught cube root by means of the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ before I ever heard of rules for algebraic multiplication or the binomial theorem. I suppose that modern algebra is what it is today, a universal language, recognizably the same in Russia and China as in New York, because it embodies the most natural* and inevitable way of representing numerical relations. Let a teacher of arithmetic, or carpentry, or what not, be encouraged to put an explanation into algebraic form whenever it is convenient to do so, and help the student to a more natural entry into algebra. If algebra produces most of the pessimists it is because it makes prematurely too ample preparations for solving complicated and difficult problems.

* Contrast the uniform interpretation of coefficients, exponents, and parentheses with the variety of notations and interpretations for derivatives in calculus, angles in geometry, and products in vectors.

Dr. E. V. Huntington says in his encyclopedia article on Symbolic Logic: "The only difficulty is to find any problems that are hard enough for this branch of the subject to attack." Everyday out-of-school experience says the same thing of all the harder parts of algebra. Dr. W. F. Osgood confesses that "the applications of algebra are to — algebra!" The problems of arithmetic and physics are for the most part absurdly simple. Algebraic analysis brings out this simplicity, and predisposes the student to study a mechanism that works so smoothly.

The casual teaching of algebra has worked well enough as I have seen it. We always have in our engineering school a good many students who are not very skillful algebraically. We used to put them into a course in that subject their first half year. Five years ago we discontinued this course. Now we get along as well as ever. We teach whatever algebra occasion demands at any time. We have got rid of the deadening effect and depressing influence of that course, and we cover more ground in two years than we could formerly, and have greatly improved the attitude of our students toward their prescribed mathematics.

In regard to changes affecting subject matter, I wish to add to the excellent proposals of Professor Babb. After reducing the emphasis on factoring, radicals, complex fractions, L.C.M., and G.C.D. in the beginners' course we can do something to bring into higher relief certain other topics.

One that I wish to call attention to is variation. In most algebras it gets very scant treatment. I think it is a subject of great fruitfulness, wholly typical of the algebraic method, and rich in applications of the every-day sort, and continually reappearing in geometry and higher mathematics.

Another is inequalities. The contrast between inequalities and equations helps break up the formalism which so easily degenerates into pessimism, and affords the student interesting opportunities for testing his own power of reasoning as to what operations are permissible. Teachers of physics treat by equations many problems in statics and in friction where a treatment by inequalities would be more adequate, more honest, and clearer to the student.

A third subject on which I would lay great stress has not even a name in our algebras. I will call it *reversal equations*. By

this term I mean *equations which contain the unknown only once,* so that to solve we have only to consider what operations have been performed upon the unknown and then to reverse each in reverse order. This class of equations has not only been unnamed but so far as I know has been nowhere recognized as an important class: yet it includes almost all the equations met with in elementary applied mathematics.* It is a class in which the analysis of the problem is peculiarly simple and natural, and in which the necessity for independent reasoning and the utility of the algebraic formulation are united in a combination which is ideal for pedagogic purposes. I consider that in calling attention to the claim of *reversal equations* to recognition as a *class* of equal importance with linears and quadratics I have made my chief contribution to algebraic progress.

Finally, if we secure the desired personality in the teacher, and if we get him into difficulties studying higher mathematics so that he can better appreciate the difficulties of his Freshmen, and if we increase the casual teaching of algebra, and redistribute the emphasis upon topics, and still have left a residue of pessimists—shall we let them go?

No doubt we waste a great deal of time trying to teach the unteachable (as philosophers do in trying to unscrew the inscrutable)—shall we merely solace ourselves with the comfort that to furnish thoughts for non-thinkers is a great achievement and worth the cost of much wasted by-product?

We cannot at present tell in advance the hopeless cases. I wish you might all have heard with me in Boston† Professor David Eugene Smith's remarkable parable in which the unpromising boy is all but prevented from ripening into a Thales, a Pythagoras, a Newton.

But if the psychologists do succeed in sifting out the "amathematicians" from among the pessimists, I hope we shall not have to remember and reproach ourselves for having forced too many complex fractions and radicals into their undigesting systems.

TUFTS COLLEGE.

* $s=\frac{1}{2}gt^2$, $A=p(1+r)^y$, $V^2=2gs$, $\sin\theta=\frac{1}{2}$, $x^{\frac{1}{4}}=3$, $V=\pi D^3/6$, etc.

† See "Some problems in the Teaching of Elementary Mathematics" in the MATHEMATICS TEACHER for March, 1913, page 161.